SOME AMAZING MAZES.

"Mazes intricate,
Eccentric, interwov'd, yet regular
Then most, when most irregular they seem."
Milton's Description of the Mystical Angelic Dance.

THE FIRST CURIOSITY.

ABOUT 1860 I cooked up a *mélange* of effects of most of the elementary principles of cyclic arithmetic; and ever since, at the end of some evening's card-play, I have occasionally exhibited it in the form of a "trick" (though there is really no trick about the phenomenon,) with the uniform result of interesting and surprising all the company, albeit their mathematical powers have ranged from a bare sufficiency for an altruistic tolerance of cards up to those of some of the mightiest mathematicians of the age, who assuredly with a little reflection could have unraveled the marvel.

The following shall describe what I do; but you, Reader, must do it too, if you are to appreciate the curiosity of the effect. So be good enough as to take two packets of playing-cards, the one consisting of a complete red suit and the other of a black suit without the king, the cards of each being arranged in regular order in the packet, so that the face-value of every card is equal to its ordinal number in the packet.

☞ N. B. *Throughout all my descriptions of manipulations of cards, it is to be understood, once for*

all, that the observance of the following STANDING RULES *is taken for granted in all cases where the contrary is not expressly directed*: Firstly, *that a pack or packet of cards* held in the hand *is, unless otherwise directed, to be held with backs up (though not, of course while they are in process of arrangement or rearrangement,) while a pile of cards* FORMED *on the table (in contradistinction to a pile placed, ready formed, on the table, as well as to rows of single cards spread upon the table,) is always to be formed with the faces displayed, and left so until they are gathered up.* Secondly, *that, whether a packet in the hand or a pile on the table be referred to, by the "ordinal, or serial, number" of a single card or of a larger division of the whole is meant its number, counting in the order of succession in the packet or pile, from the card or other part at the* BACK *of the packet or at the* BOTTOM *of the pile as* "Number 1," *to the card or other part at the* FACE *of the packet or the* TOP *of the pile; the ordinal or serial number of this last being equal to the cardinal number of cards (or larger divisions* COUNTED,*) in the whole packet or pile; and the few exceptions to this rule will be noted as they occur;* Thirdly,*that by the "face-value" is meant the number of pips on a plain card, the ace counting as one; while, of the picture-cards, the knave, for which* J *will usually be written, will count as eleven, the queen, or* Q, *as twelve, and the king,* K, *either as thirteen or as the zero of the next suit;* and Fourthly, *that when a number of piles that have been formed upon the table by dealing out the cards, are to be gathered up, the uniform manner of doing so is to be as follows: The first pile to be taken (which pile this is to be will appear in due time,) is to be grasped as a whole and placed* (faces up,) *upon the pile that is to be taken next. Then those two piles are to be grasped as a whole, and placed* (faces up,) *upon the pile that is next to be taken; and so on, until all the*

piles have been gathered up; when, in accordance with the first Standing Rule, the whole packet is to be turned back up. And note, by the way, that in consequence of the manner in which the piles are gathered, each, after the first, being placed at the back of those already taken, while in observance of the second Standing Rule, we always count places in a packet from the back of it, it follows that the last pile taken will be the first in the regathered packet, while the first taken will become the last, and all the others in the same complementary way, the ordinal numbers of their gathering and those of their places in the regathered packet adding up to one more than the total number of piles.

Of course, while the red packet and the black packet are getting arranged so that the face-value of each card shall also be its ordinal, or serial, number in the packet, the cards must needs be held faces up. But as soon as they have been arranged, the packet of thirteen cards is to be laid on the table, *back up.* You then deal,—for, let me repeat it, Reader, by the inexorable laws of psychology, if you do not actually take cards, (and the U. S. Playing-Card Company's "Fauntleroy" playing cards are the most suitable, although any that run smoothly will do,) and actually go through the processes, the whole description can mean nothing to you;—*you* deal, then, the twelve black cards, one by one, into two piles, the first card being turned to form the bottom of the first pile, the second that of the second pile (on the right hand of the first pile,) the third card going on the first pile again, the fourth on the second, and every following card being placed immediately upon the card whose ordinal, or serial, number in the packet before the deal was two lower than the former's ordinal, or serial, number then was. *The last card, however, is to be exceptionally treated.* Instead of being placed on the top of the second pile according to the rule just given, it is

to be placed on the table, face up, and apart from the other cards, to make the bottom card of an isolated pile, to be called the *"discard pile"*; while, in place of it, the first card of the pile of cards of the red suit, which card will, of course, be the ace, is to be placed face up on the top of the second of the two piles formed by the dealing, where that discarded card would naturally have gone. Now you gather up these two piles by grasping the first, or left-hand, pile, placing it, face up, upon the second, or right-hand, pile, and taking up the two together; and you then at once turn the packet back up in compliance with the first standing rule. This whole operation of *firstly,* dealing out into two piles the packet that was at first entirely composed of black cards; but *secondly,* placing the last card, face up, on the discard pile, and *thirdly,* substituting for it the card then at the top of the pile of red cards, by placing this latter, face up, upon the top of the second pile of the deal, and then, *fourthly,* putting the left-hand, or first, pile of the deal, face up, upon the second, and having taken up the whole packet, turning it with its back up,—this whole quadripartite operation, I say, is to be performed, in all, twelve times in succession. My statement that in this operation the last card is treated *exceptionally* was quite correct, since its treatment made an exception to the rule of placing each card on the card that before the deal came two places in advance of it in the packet. Had I said it was treated *irregularly,* I should have written very carelessly, since it is just one of those cases in which a violation of a regularity of a low order establishes a regularity of a much higher order, (if John Milton knew the meaning of the word "regular,")—a pronouncement which must be left for the issue of the performance to ratify; and you shall see, Reader, that the event will ratify it with striking emphasis. Already, we begin to see some regularity in the process, since each of

the twelve cards placed on the discard-pile in the twelve performances of the quadripartite operation is seen to belong to the black-suit; so that the packet held in the hand and dealt out, from being originally entirely black, has now become entirely red. Having placed the red king upon the face of this packet, you now lay down the latter in order to have your hands free to manipulate the discard-pile. Holding this discard-pile as the first standing rule directs, you take the cards singly from the top and range them, one by one, from left to right, in a row upon the table, with their backs up. The length of the table from left to right ought to be double that of the row; and this is one of the reasons for preferring cards of a small size. To guard against any mistake, you may take a peek at the seventh card, to make sure that it is the ace, as it should be. The row being formed, I remark to the company, as you should do in substance, that I reserve the right to move as many of these black cards as I please, at any and all times, from one end of the row to the other; but that beyond doing that, I renounce all right to disarrange those cards. Then, taking up the red cards, and holding the packet with its back up, I (and so must you,) request any person to cut it. When he does so, you place the cards he leaves in your hand at the back of the partial packet he removes. This is my proceeding, and must be yours. You then ask some person to say into how many piles (less than thirteen,) the red cards shall be dealt. When he has prescribed the number of piles, you are to hold the packet of red cards back up, and deal cards one by one from the back of it, placing each card on the table face up, and each to the right of the last card dealt. When you have dealt out enough to form the bottom cards of piles to the number commanded, you return to the extreme left-hand pile, *which you are to imagine as lying next to, and to the right of, the extreme right-hand pile,*—as in fact it would come

next in clockwise order, if the row were bent down at the ends in the manner shown in Fig. 1, where the piles (here supposed to be eight in all,) are numbered in the order in which their bottom cards are laid down. Indeed, when more than seven piles are ordered, it is not a bad plan actually to arrange them so. So, counting the piles round and round, whether you place them in a circle or not, you place each card on the pile that comes clockwise next after, or to the right of the pile upon which the card next before it was placed (regulating your imagination as above stated,) and so you continue until you have dealt out the whole packet of thirteen cards. You now proceed to gather up the piles according to the Fourth Standing Rule.

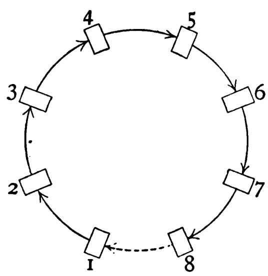

Fig. 1.

That rule, however, does not determine the order of succession in which the piles are to be taken up. I will now give the rule for this. It applies to the dealing of any prime number of cards, or of any number of cards one less than a prime number, into any number of piles less than that prime number. It happens that that form of statement of this rule which is decidedly the most convenient when the number of piles does not exceed seven, as well as when the whole number of cards differs by less than three from some multiple of the number of piles, becomes quite confusing in other cases. A slight modification of it which I will give as a second form of the rule, sometimes greatly mitigates the inconvenience; and it will be well to acquaint yourself with it. But for the most part, when the first form threatens to be confusing, it will be best to resort to that form of the rule which I describe as the third.

For the purpose of this "first curiosity" (indeed, in every case where a prime number of real cards are dealt out,) it

matters not what pile you take up first. But in certain cases we shall have occasion to deal out into piles a number of cards, such as 52, which is one less than a prime number. In such case, it will be necessary to add *an imaginary card* to the pack, since a real card would interfere with certain operations. Now imaginary cards, if allowed to get mixed in with real ones, are liable to get lost. Consequently, in such cases, we have to keep the imaginary card constantly at the face of the pack by taking up first the pile on which it is imagined to fall, that is, the pile next to the right of the one on which the last real card falls. I now proceed to state, in its three forms, the rule for determining what pile is to be taken up next after any given pile that has just been taken. It is assumed that the whole pack of cards dealt consists of a prime number of cards; but, of these cards, the last may be an imaginary one, provided the pile on which it is imagined regularly to fall be taken up first.

First Form of the Rule. Count from the place of the extreme right-hand pile, as zero, either way round, clockwise or counterclockwise,—preferably in the shortest way, —to the place of the pile on which the last card, real or imaginary, fell. Then, counting the original places of piles, whether the piles themselves still remain in those places or have already been picked up, from the place of the pile last taken, in the same direction, up to the same number, you will reach the place of the next pile to be taken.

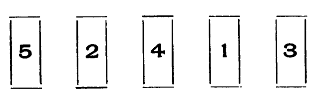

Fig. 2.

Example. If 13 cards are dealt into five piles, the 13th card will fall on the second pile from the extreme right-hand pile going round counter-clockwise. Supposing, then,

that the first pile taken is the right-handmost but one, they are all to be taken in the order marked in Fig. 2.

Second Form. Proceed as in the first form of the rule until you have repassed the place of the first pile taken. You will then always find that the place of the last pile taken is nearer to that of some pile, P, previously taken, than it is to the place of that taken immediately before it. Then, the next pile to be taken will be in the same relation of places to the pile taken next after the pile P.

Example. Let 13 cards be dealt into 9 piles. Then the last card will fall on the pile removed 4 places clockwise from the extreme right-hand pile. Then, when you have removed four piles according to the first form of the rule, you will at once perceive, as shown in Fig. 3, (where

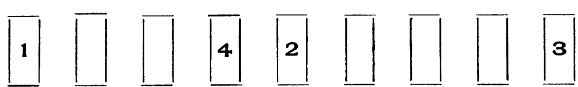

Fig. 3.

it is assumed that the extreme left-hand pile was the one to be taken up first,) that for the rest of the regathering, you have simply to take the pile that stands immediately to the left of the place of the last previous removal but one.

Third Form. In this form of the rule vacant places are not counted, but only the remaining piles, which is sometimes much less confusing. It is requisite, however, carefully to note the place of the pile first taken. You begin as in the first form of the rule; but every time you pass over the place whence the first pile was removed, you diminish the number of your count by one, beginning with the count then in progress; and you adhere to this number until you pass the same place again, and consequently again diminish the number of your count, which will thus ultimately be reduced to one, when you will take every pile you come to.

Example. Let a pack of 52 cards be dealt into 22 piles. The first pile taken up must be the one upon which the imaginary 53d card falls. It is assumed that, before the deal the cards were arranged in suits in the order ♢ ♠ ♡ ♣ and in each suit in the order of their face-values. Then the different columns of Fig. 4 show the cards at the tops of the different piles while the different horizontal rows

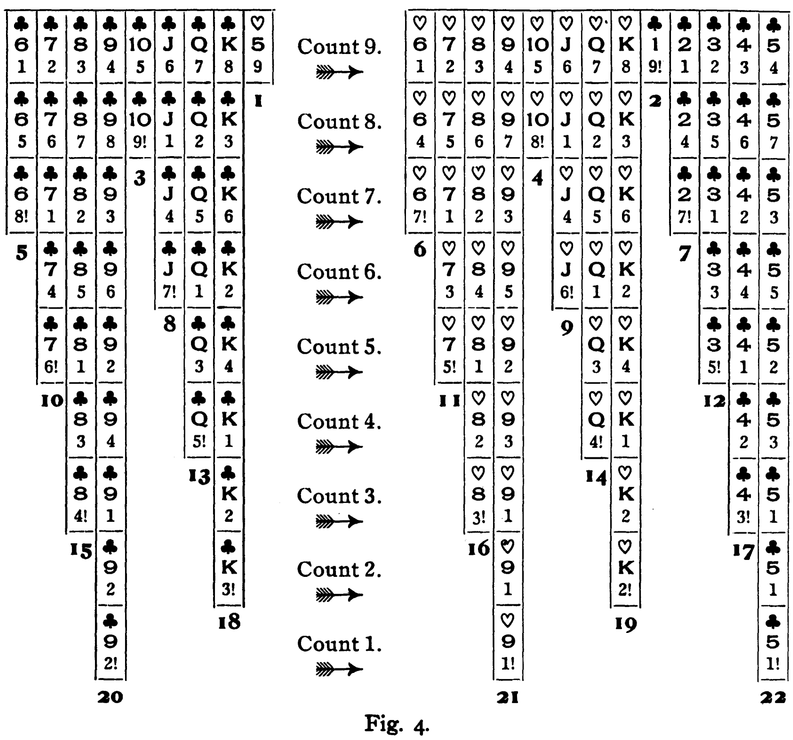

Fig. 4.

show what piles remain, just before you come to count the left-hand-most of the remaining piles, as your countings successively pass through the whole row of piles. The gap between the columns just after the place where the imaginary card is supposed to have fallen, contains the direction thereafter to diminish by one the number of piles you count. Beneath the designations of the top cards are small type

numbers which are the numbers in your different countings through the row of piles; and the last number in each count is followed by a note of admiration that is to be understood as a command to gather up that pile. Beneath it is a heavy faced number, which is the ordinal number of that removal.

I hate to bore readers who are capable of exact thought with redundancies; but others often deploy such brilliant talents in not understanding the plainest statements that have no familiar jingle, that I must beg my more active-

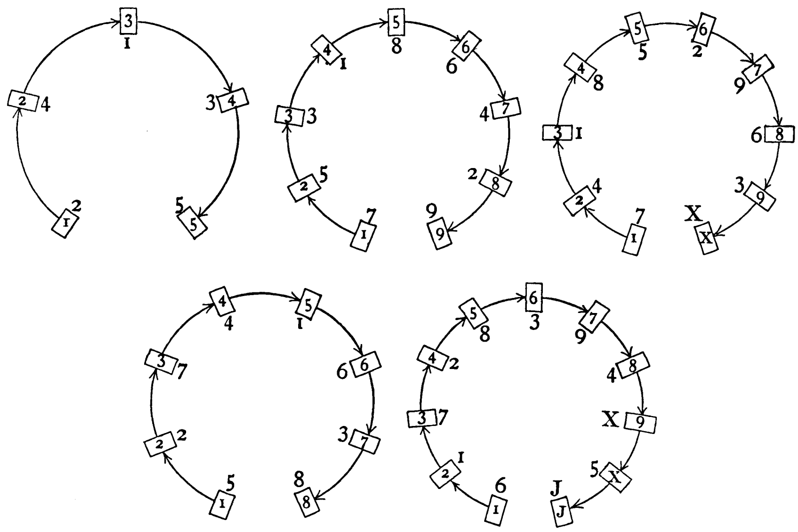

Fig. 5.

minded readers to have patience under the infliction while I exhibit in Fig. 5 the orders in which 5, 8, 9, 10, and 11 piles formed by dealing 13 cards are to be taken up.

When the red cards have thus been regathered, you again hold out the packet to somebody to cut, and again request somebody to say into how many piles they shall be dealt "in order that the mixing may be as thorough as it may." You follow his directions, and regather the piles according to the same rule as before. If your com-

pany is not too intelligent, you might venture to ask somebody, before you regather the piles, to say what pile you shall take up first; but this will be presuming a good deal upon the stupidity of the company; for an inference might be drawn which would go far toward destroying the surprise of the result. Nothing absolutely prevents the cards from being cut and dealt any number of times. When the number of piles for the last dealing has been given out, you will have to ascertain what transposition of the black cards is required. There are three alternative ways of doing this, which I proceed to describe.

The best way is to multiply together the numbers of piles of the different dealings of the red cards, subtracting from each product the highest multiple of 13, if there be any, that is less than that product. The result is the cyclical product. By "the different dealings," you here naturally understand those that have taken place since the last shifting of the black row. If a wrong shift has been made, the simplest way to correct it, after new cuttings and dealings, is to resort to a peep at the black ace, and to determining where it ought to be in the third way explained below.

Thus, if the red cards have been dealt into 5 piles and into 3 piles, since 3 times 5 make 15, and 15—13=2, the cyclical product is 2. You now proceed to ascertain how many times 1 has to be cyclically doubled to make that cyclical product. But if 6 doublings do not give it,—which six doublings will give

1 doubling, twice 1 are 2,
2 doublings, twice 2 are 4,
3 doublings, twice 4 are 8,
4 doublings, twice 8 less 13 make 3,
5 doublings, twice 3 are 6,
6 doublings, twice 6 are Q,—

I say if none of the first six doublings gives the cyclical product of the numbers of piles in the dealings, you resort to successive cyclical halvings of 1. The cyclical half of an even number is the simple half; but to get the cyclical half of an odd number, add 7 to half of one less than that number. Thus,

The cyclical half of 1 is $(0 \div 2) + 7 = 7$;
" " " " 7 is $(6 \div 2) + 7 = X$;
" " " " X is 5;
" " " " 5 is $(4 \div 2) + 7 = 9$;
" " " " 9 is $(8 \div 2) + 7 = J$;
" " " " J is $(X \div 2) + 7 = Q$.

If the cyclical product of the numbers of piles in the dealings is one of the first six results of doubling one, you will have (when the time comes,) to bring one card from the right-hand end of the row of black cards to the left-hand end for each such doubling. Thus, if the red cards have twice been dealt into 4 piles, four cards must be brought from the right end to the left end of the row of black cards. For $4 \times 4 - 13 = 3$ and $1 \times 2^4 - 13 = 3$. But if that cyclical product is one of the first six results of successive cyclical halvinngs of one, one card must be carried from the left to the right end of the row of black cards for every halving. Thus, if the red cards have been dealt into 6 and into 8 piles, 4 black cards must be carried from the left-hand end of the row to the right-hand end of the row. $6 \times 8 - 3 \times 13 = 9$ and it takes 4 cyclical halvings to give 9. If the product of the numbers of piles in the dealings is one more than a multiple of 13, the row of black cards is to remain unshifted.

The second way of determining how the black cards are to be transposed is simply, during the last of the dealings, to note what card is laid upon the king. The face-value of this card is the ordinal, or serial place in the row,

counting from the left-hand extremity of it, which the ace must be brought to occupy. Now if you remember, as you always ought to do, where the ace is in the row, you will know how many cards to carry from one end to the other so as to bring the ace into that place. But if in the last dealing the king happens to fall at the top of one of the piles, two lines of conduct are open to you. One would be, in regathering the piles, by a pretended awkwardness in taking up the pile that is to be taken next before the one that the king heads, at first to leave its bottom card on the table, so as to get a glimpse of it before you take it up, as you would regularly have done at first; and if the king should happen to be the last card dealt, the card at the back of the packet would be the one for you to get sight of, by a similar imitation blunder. In either case, the card you so aim to get sight of would show the right place for the ace in the row. But if you doubt your ability to be gracefully awkward, it always remains open to you to ask to have the red packet cut again and a number of piles for a new deal to be ordered.

The third way of determining the proper transposition of the black cards is a slight modification of the second. It consists in looking at the card whose back is against the face of the king, when you come to cut the red packet so as to bring the king to the face. [Any practical psychologist, such as a prestigiator must be, can, with the utmost ease look for the card he wants to see, and can inspect it without detection.]

But whichever of these methods you employ, you should not touch the row of black cards until the red cards, having been regathered after the last dealing, you have said something like this: "Now I think that all these dealings and cuttings and exchanges of the last cards have sufficiently mixed up the red cards to give a certain interest to the fact that I am going to show you; namely, that

this row of black cards form an index showing where any red card you would like to see is to be found in the red pack. But since there is no black king in the row, of course the place of the red king cannot be indicated; and for that reason, I shall just cut the pack of red cards so as to bring the king to the face of it, and so render any searching for that card needless." You then cut the red cards. That speech is quite important as restraining the minds of the company from reflecting upon the relation between the effect of your cutting and that of theirs. Without much pause you go on to say that you shall leave the row of black cards just as they are, simply putting so many of them from one end of the row to the other. You now ask some one, "Now, what red card would you like to find?" On his naming the face-value of a card, you begin at the left-hand end of the row of black cards and count them aloud and deliberately, pointing to each one as you count it, until you come to the ordinal number which equals the face value of the red card called for; and in case that card is the knave or queen, you call "knave" instead of "eleven" on pointing at the eleventh card, and "queen" on pointing at the last card. When you come to call the number that equals that of the red card called for, you turn the card you are pointing at face up. Suppose it is the six, for example. Then you say, naming the card called for, that that card will be the sixth; or if the card turned up was the knave, you say that the card called for will be "in the knave-place," and so in other cases. You then take up the red packet, and counting them out, aloud and deliberately, from one hand to the other, and from the back toward the face of the packet, when you come to the number that equals the face-value of the black card turned, you turn over this card as soon as you have counted it, and lo! it will be the card called for.

The company never fail to desire to see the thing done

again; and on their expressing this wish, after impressing on your memory the present place of the black ace, you have only to hold out the red cards to be cut again, and you again go through the rest of the performance, now abbreviating it by having the cards dealt only once. The third time you do it, since you will now have given them the enjoyment of their little astonishment, there will no longer be any reason for not asking somebody to say what pile you shall take up first, although that will soon lead to their seeing that all the cuttings are entirely nugatory. Still they will not thoroughly understand the phenomenon.

If you wish for an explanation of it, the wish shows that you are not thoroughly grounded in cyclic arithmetic, and that you consequently still have before you the delight of assimilating the first three *Abschnitte* (for that matter the first hundred pages would suffice to reveal the foundations of the present mystery; but I confess I do not particularly admire the first Abschnitt) of Dedekind's lucid and elegant redaction of the unerring Lejeune-Dirichlet's "*Vorlesungen über Zahlentheorie.*" But, perhaps, on another occasion I will myself give a little essay on the subject, "adapted to the meanest capacity," as some of the books of my boyhood used, not too respectfully, to express it.

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